



**CSE 417**  
**Algorithms and Complexity**  
 Autumn 2024  
 Lecture 29  
 NP-Completeness and Beyond

## Announcements

- Exam practice problems on course homepage
- Final Exam: Monday, December 9, 8:30 AM

|            |                            |
|------------|----------------------------|
| Mon, Dec 2 | NP-Completeness            |
| Wed, Dec 4 | NP-Completeness            |
| Fri, Dec 6 | NP-Completeness and Beyond |
| Mon, Dec 9 | Final Exam                 |

- This is my last lecture

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## NP-Completeness Proofs

- Prove that problem X is NP-Complete
  - Show that X is in NP (usually easy)
  - Pick a known NP complete problem Y
  - Show  $Y \leq_P X$

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## Reducibility Among Combinatorial Problems

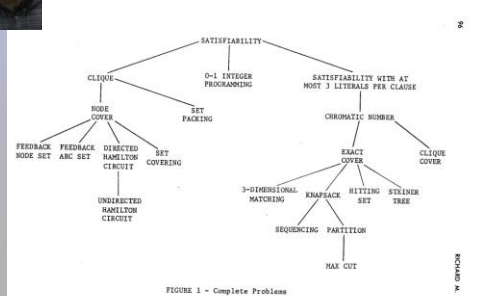
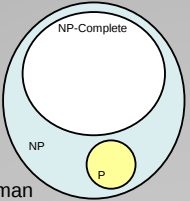


FIGURE 1 - Complete Problems

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## Populating the NP-Completeness Universe



- Circuit Sat  $\leq_P$  3-SAT
- 3-SAT  $\leq_P$  Independent Set
- 3-SAT  $\leq_P$  Vertex Cover
- Independent Set  $\leq_P$  Clique
- 3-SAT  $\leq_P$  Hamiltonian Circuit
- Hamiltonian Circuit  $\leq_P$  Traveling Salesman
- 3-SAT  $\leq_P$  Integer Linear Programming
- 3-SAT  $\leq_P$  Graph Coloring
- 3-SAT  $\leq_P$  Subset Sum
- Subset Sum  $\leq_P$  Scheduling with Release times and deadlines

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## Coping with NP-Completeness

- Approximation Algorithms
- Exact solution via Branch and Bound
- Local Search

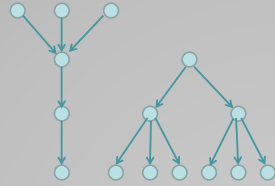


I can't find an efficient algorithm, but neither can all these famous people.

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## Multiprocessor Scheduling

- Unit execution tasks
- Precedence graph
- K-Processors
- Polynomial time for  $k=2$
- Open for  $k = \text{constant}$
- NP-complete is  $k$  is part of the problem



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## Highest level first is 2-Optimal

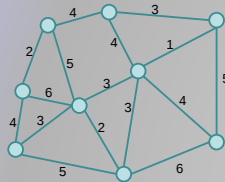
Choose  $k$  items on the highest level

Claim: number of rounds is at least twice the optimal.

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## Christofides TSP Algorithm

- Undirected graph satisfying triangle inequality

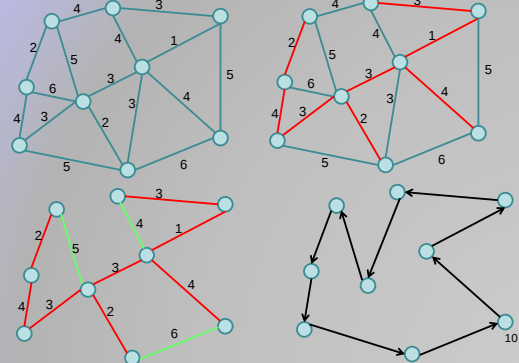


1. Find MST
2. Add additional edges so that all vertices have even degree
3. Build Eulerian Tour

3/2 Approximation

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## Christofides Algorithm



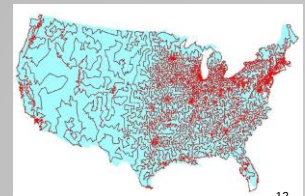
## Branch and Bound

- Brute force search – tree of all possible solutions
- Branch and bound – compute a lower bound on all possible extensions
  - Prune sub-trees that cannot be better than optimal

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## Branch and Bound for TSP

- Enumerate all possible paths
- Lower bound, Current path cost plus MST of remaining points
- Euclidean TSP
  - Points on the plane with Euclidean Distance
  - Sample data set: State Capitals



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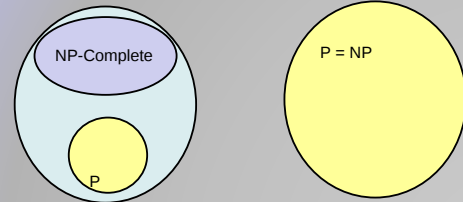
## Local Optimization

- Improve an optimization problem by local improvement
  - Neighborhood structure on solutions
  - Travelling Salesman 2-Opt (or K-Opt)
  - Independent Set Local Replacement

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## What we don't know

- P vs. NP

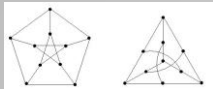


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## If $P \neq NP$ , is there anything in between

- Yes, Ladner [1975]
- Problems not known to be in P or NP Complete
  - Factorization
  - Discrete Log
  - Graph Isomorphism

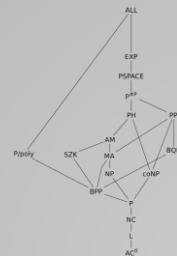
Solve  $g^k = b$  over a finite group



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## Complexity Theory

- Computational requirements to recognize languages
- Models of Computation
- Resources
- Hierarchies
- [//complexityzoo.net](http://complexityzoo.net)



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## Time complexity

- P: (Deterministic) Polynomial Time
- NP: Non-deterministic Polynomial Time
- EXP: Exponential Time

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## Space Complexity

- Amount of Space (Exclusive of Input)
- L: Logspace, problems that can be solved in  $O(\log n)$  space for input of size  $n$ 
  - Related to Parallel Complexity
- PSPACE, problems that can be required in a polynomial amount of space

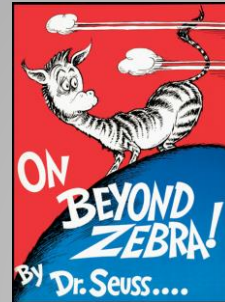
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## Other types of computation

- Randomization
  - Can you solve problems faster with a random number generator?
- Quantum Computers
  - Can you solve problems faster if you have quantum bits which allow superposition?
    - Probably yes: Shor's Integer Factoring algorithm

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## So what is beyond NP?



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## NP vs. Co-NP

- Given a Boolean formula, is it true for some choice of inputs
- Given a Boolean formula, is it true for all choices of inputs

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## Problems beyond NP

- Exact TSP, Given a graph with edge lengths and an integer K, does the minimum tour have length K
- Minimum circuit, Given a circuit C, is it true that there is no smaller circuit that computes the same function as C

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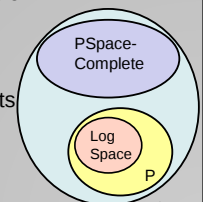
## Polynomial Hierarchy

- Level 1
  - $\exists X_1 \Phi(X_1), \forall X_1 \Phi(X_1)$
- Level 2
  - $\forall X_1 \exists X_2 \Phi(X_1, X_2), \exists X_1 \forall X_2 \Phi(X_1, X_2)$
- Level 3
  - $\forall X_1 \exists X_2 \forall X_3 \Phi(X_1, X_2, X_3), \exists X_1 \forall X_2 \exists X_3 \Phi(X_1, X_2, X_3)$

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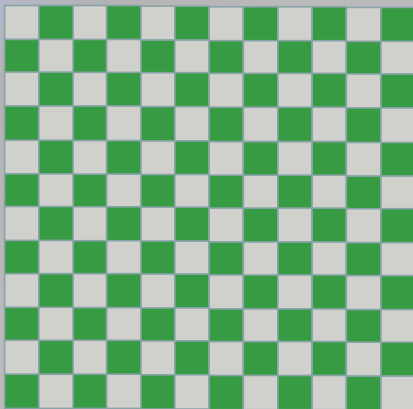
## Polynomial Space

- Quantified Boolean Expressions
  - $\exists X_1 \forall X_2 \exists X_3 \dots \exists X_{n-1} \forall X_n \Phi(X_1, X_2, X_3 \dots X_{n-1} X_n)$
- Space bounded games
  - Competitive Facility Location Problem
  - N x N Chess
- Counting problems
  - The number of Hamiltonian Circuits



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## N X N Chess



## Even Harder Problems

```
public int[] RecolorSwap(int[] coloring) {
    int k = maxColor(coloring);

    for (int v = 0; v < nVertices; v++) {
        if (coloring[v] == k) {
            int[] nbdColorCount = ColorCount(v, k, coloring);
            List<Edge> edges1 = vertices[v].Edges;

            foreach (Edge e1 in edges1) {
                int w = e1.Head;
                if (nbdColorCount[coloring[w]] == 1)
                    if (RecolorSwap(v, w, k, coloring))
                        break;
            }
        }
    }
    return coloring;
}
```

Is this code correct?

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## Halting Problem

- Given a program P that does not take any inputs, does P eventually exit?

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## Impossibility of solving the Halting Problem

Suppose  $\text{Halt}(P)$  returns true if P halts, and false otherwise

Consider the program G:

What is  $\text{Halt}(G)$ ?

```
bool G {
    if (Halt(G)){
        while (true) ;
    }
    else {
        return true;
    }
}
```

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## Undecidable Problems

- The Halting Problem is undecidable
- Impossible problems are hard . . .

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