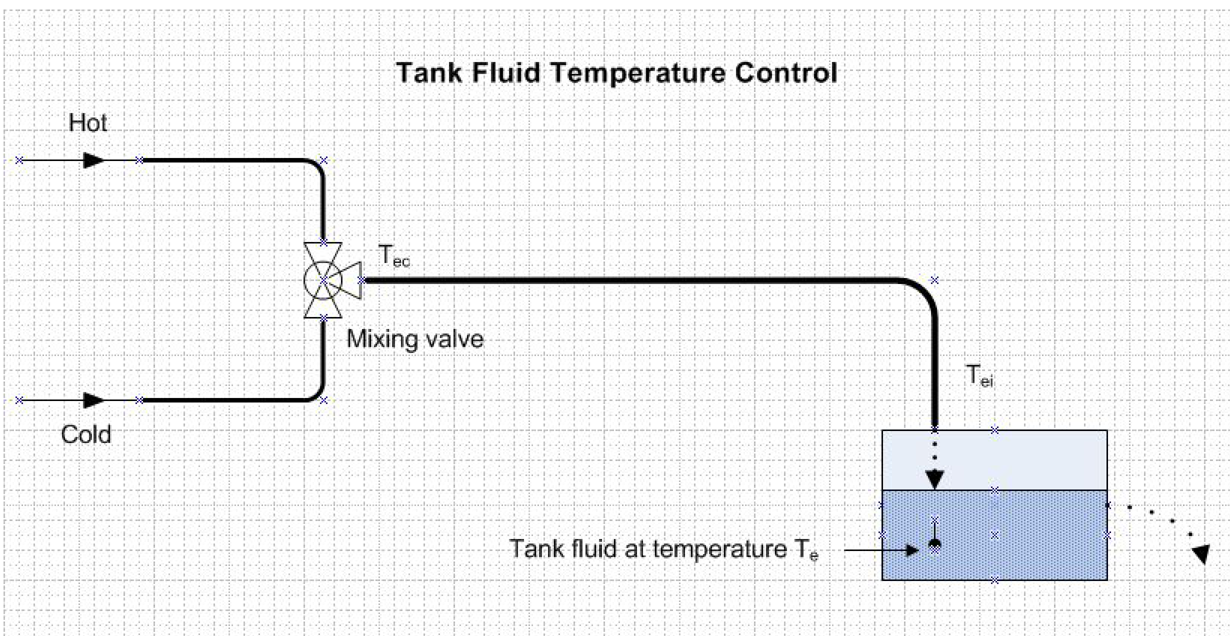
# MPC Case Studies

## Temperature Control of Fluid in a Tank

- The temperature of the fluid contained in a tank with a constant flow rate in and out is to be controlled
- The control variable is the temperature of the incoming fluid which is adjusted by a mixing valve that regulates the relative amounts of hot and cold fluid supply, as indicated in the diagram below.




- Fluid temperature inside the tank is governed by the following first-order differential equation:

$$\dot{T}_e = \frac{1}{cM}(q_i - q_o)$$

where

$$
\begin{aligned}
c &\triangleq \text{ specific heat of fluid} \\
M &\triangleq \text{fluid mass in tank} \\
T_e &\triangleq \text{tank temperature} \\
q_i &= c\dot{m}_i T_{ei} \\
q_o &= c\dot{m}_o T_e \\
\dot{m} &\triangleq \text{mass flow rate}
\end{aligned}
$$

- The temperature at the tank input at time $t$ is the control temperature, $T_{ec}$
- A fluid transport delay of $\tau_d$ seconds is present between the mixing valve outlet and the tank input:

$$T_{ei}(t) = T_{ec}(t - \tau_d)$$

- Substituting we get

$$
\begin{aligned}
\dot{T}_e &= \frac{1}{cM} q_i - \frac{1}{cM} q_o \\
&= \frac{1}{cM} c\dot{m}_i T_{ei} - \frac{1}{cM} c\dot{m}_o T_e \\
&= \frac{\dot{m}_i}{M} T_{ei} - \frac{\dot{m}_o}{M} T_e
\end{aligned}
$$

which allows us to write

$$\dot{T}_e + \left(\frac{\dot{m}_o}{M}\right) T_e = \left(\frac{\dot{m}_i}{M}\right) T_{ec}(t - \tau_d)$$

- Let

$$a = \frac{\dot{m}_o}{M} = \frac{\dot{m}_i}{M} = \frac{\dot{m}}{M}$$

and we have

$$\dot{T}_e(t) + aT_e(t) = aT_{ec}(t - \tau_d)$$

- We can now generate a transfer function between the output variable $T_e(s)$ and the input $T_{ec}(s)$ :

$$\frac{T_e(s)}{T_{ec}(s)} = \frac{e^{-\tau_d s}}{s/a + 1} = G(s)$$

- Converting to discrete-time we generate the $\mathcal{Z}$-Transform, assuming a zero-order hold,

$$\mathcal{Z}\left\{\frac{1 - e^{-Ts}}{s} \cdot \frac{e^{-\tau_d s}}{s/a + 1}\right\}$$

- Assume

$$\begin{aligned}\tau_d &= \ell T - mT, \quad 0 < m \leq 1 \\ &= (\ell - m)T\end{aligned}$$

Then,

$$\begin{aligned}G(z) &= \mathcal{Z}\left\{\frac{1 - e^{-Ts}}{s} \cdot \frac{e^{-\ell Ts}e^{mTs}}{s/a + 1}\right\} \\ &= (1 - z^{-1})z^{-\ell} \cdot \mathcal{Z}\left\{\frac{e^{mTs}}{s(s/a + 1)}\right\} \\ &= (1 - z^{-1})z^{-\ell} \cdot \mathcal{Z}\left\{\frac{e^{mTs}}{s} - \frac{e^{mTs}}{s + a}\right\} \\ &= (\frac{z - 1}{z}) \cdot \left(\frac{1}{z^{\ell}}\right) \cdot \left(\frac{z}{z - 1} - \frac{e^{-amT}}{z - e^{-amT}}\right) \\ &= \frac{(1 - e^{-amT})z + e^{-amT} - e^{-aT}}{z - e^{-aT}}\end{aligned}$$

- This gives

$$G(z) = \frac{1 - e^{-amT}}{z^{\ell}} \cdot \frac{z + \alpha}{z - e^{-aT}}$$

where we've defined

$$\alpha = \frac{e^{-amT} - e^{-aT}}{1 - e^{-amT}}$$

CASE 1

- Let $\dot{m} = 1000\,\mathrm{kg/s}$ and $M = 1000\,\mathrm{kg}$
  - This means that a volume equal to the entire content of the tank (i.e., $1.0\,\mathrm{m}^3$) will flow in and out each second
  - This gives $a = \dot{m}/M = 1$
  - Further assume that the sampling time $T = 1\,\mathrm{s}$ and the transport time delay is $\tau_d = 1.5\,\mathrm{s}$; which means $\ell = 2.0\,\mathrm{s}$ and $m = 0.5\,\mathrm{s}$.

- Substituting into the expression for $G(z)$ above yields

$$G_1(z) = \frac{Y(z)}{U(z)} = \frac{(.3935) \cdot (z + 0.6065)}{z^2(z - 0.3679)} = \frac{.3935z^{-2} + .2387z^{-3}}{1 - .3679z^{-1}}$$

- In difference equation form this gives,

$$y(k) = 0.3679y(k-1) + 0.3935u(k-2) + 0.2387u(k-3)$$

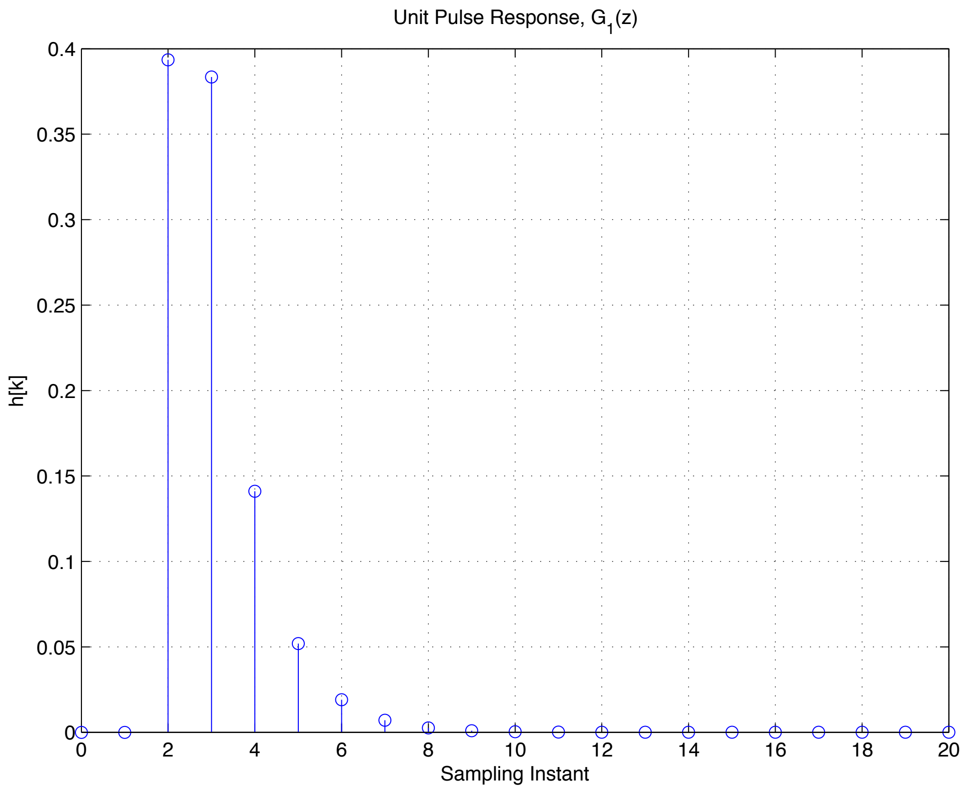
or in general terms

$$y(k) = e^{-aT}y(k-1) + (1 - e^{-amT})u(k-2) + (e^{-amT} - e^{-aT})u(k-3)$$

- Note that the zero location varies considerably as $m$ varies throughout its range:

$$\alpha \rightarrow 0 \text{ as } m \rightarrow 1$$
$$\alpha \rightarrow \infty \text{ as } m \rightarrow 0$$

- Let's now examine the unit pulse response for Case 1

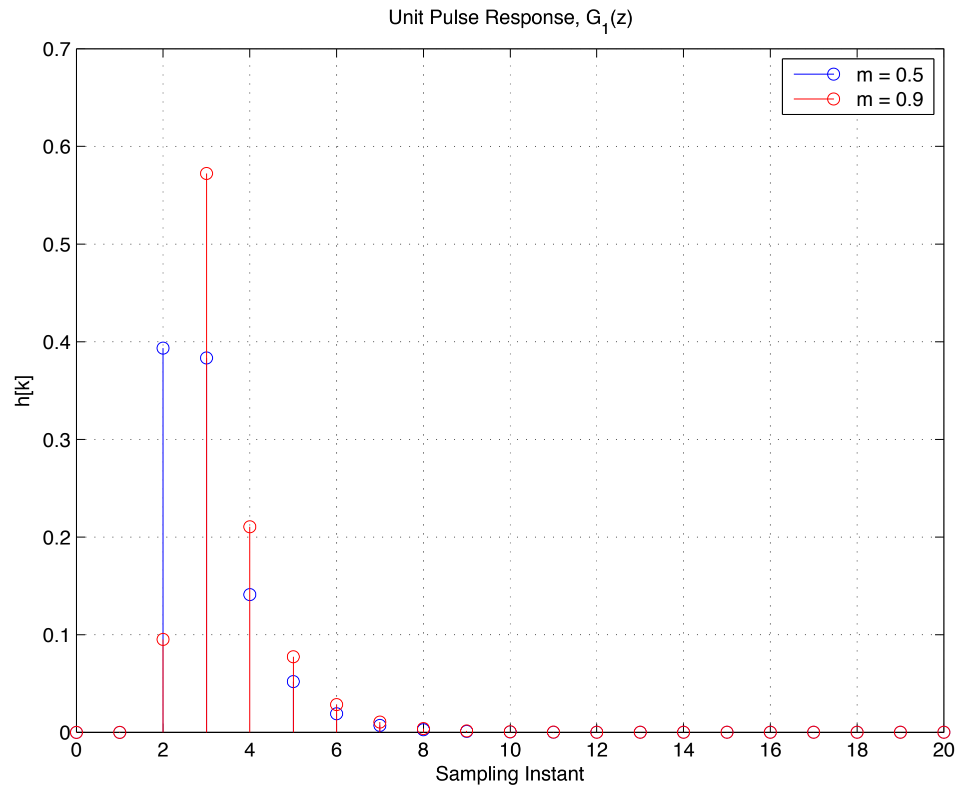



- Now if we assume a time delay $\tau_d = 1.9\,\mathrm{s}$ , then this gives $\ell = 2.0\,\mathrm{s}$ and $m = 0.1\,\mathrm{s}$
- The resulting transfer function is

$$G_2(z) = \frac{(.0952) \cdot (z + 5.6425)}{z^2(z - 0.3679)}$$

indicating a non-minimum phase zero.

  – We plot both impulse responses below and note the different dynamics

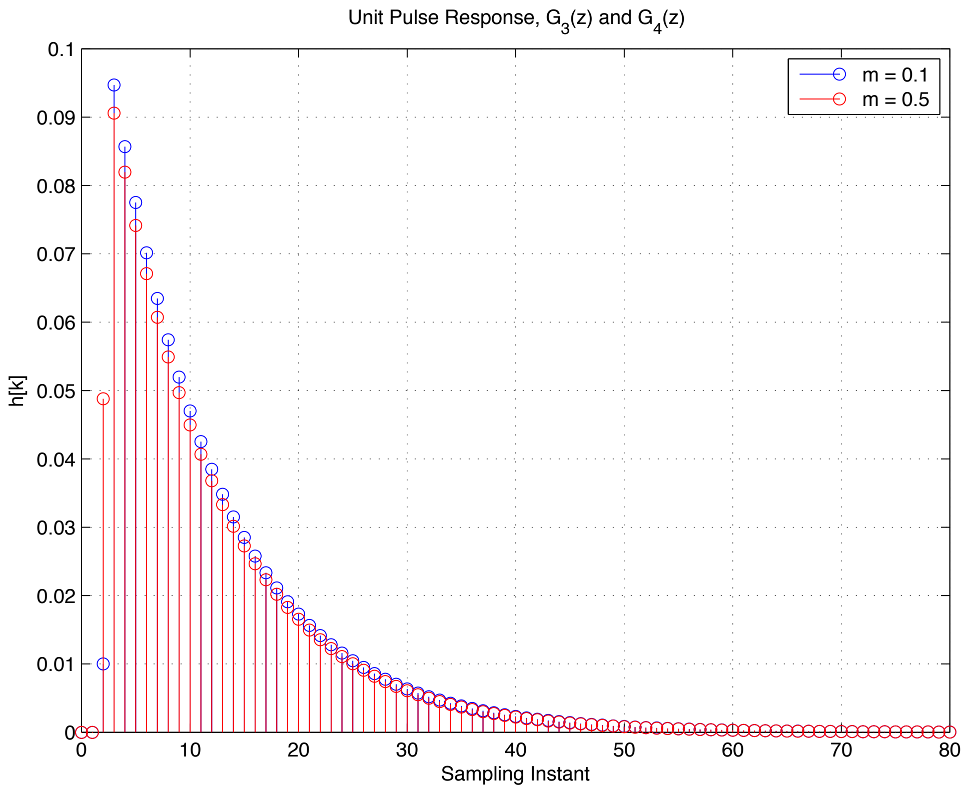



- More realistically, we let the flow rate $\dot{m} = 100\,\mathrm{kg/s}$ which gives a ratio $\dot{m}/M = 0.1$
- Keeping the sampling interval the same, we generate two new transfer functions for $m = .1$ and $m = .5$ respectively,

$$G_3(z) = \frac{(.01)(z + 8.5639)}{z^2(z - 0.9048)}$$

$$G_4(z) = \frac{(.0488)(z + .9512)}{z^2(z - .9048)}$$

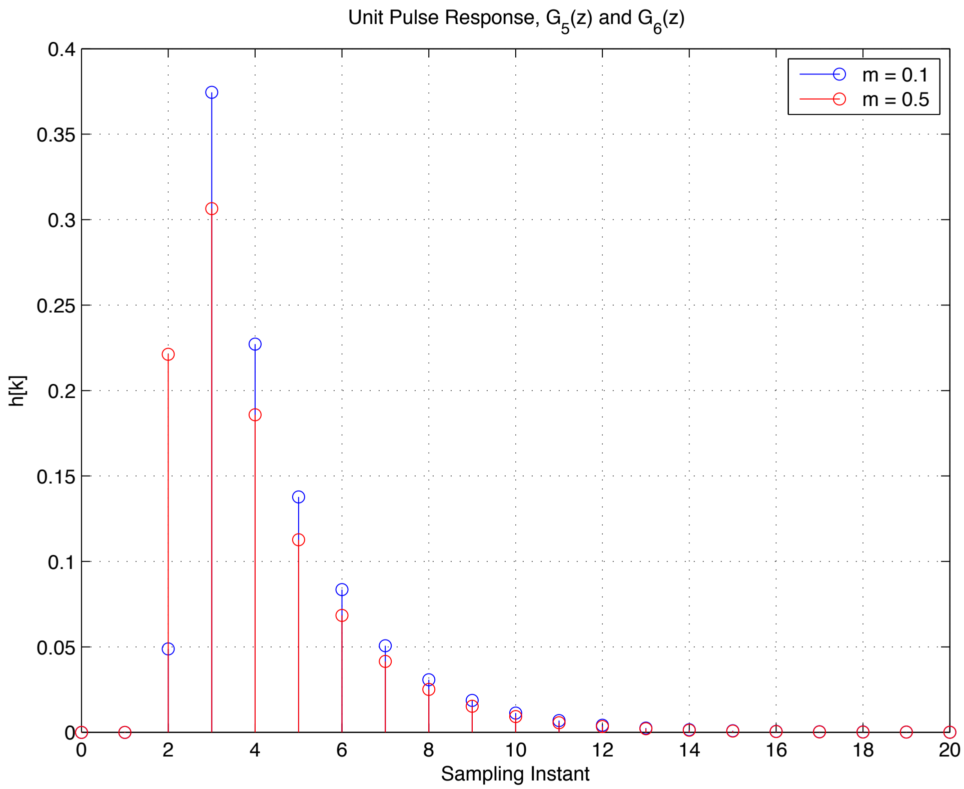
  – The corresponding impulse responses are plotted below



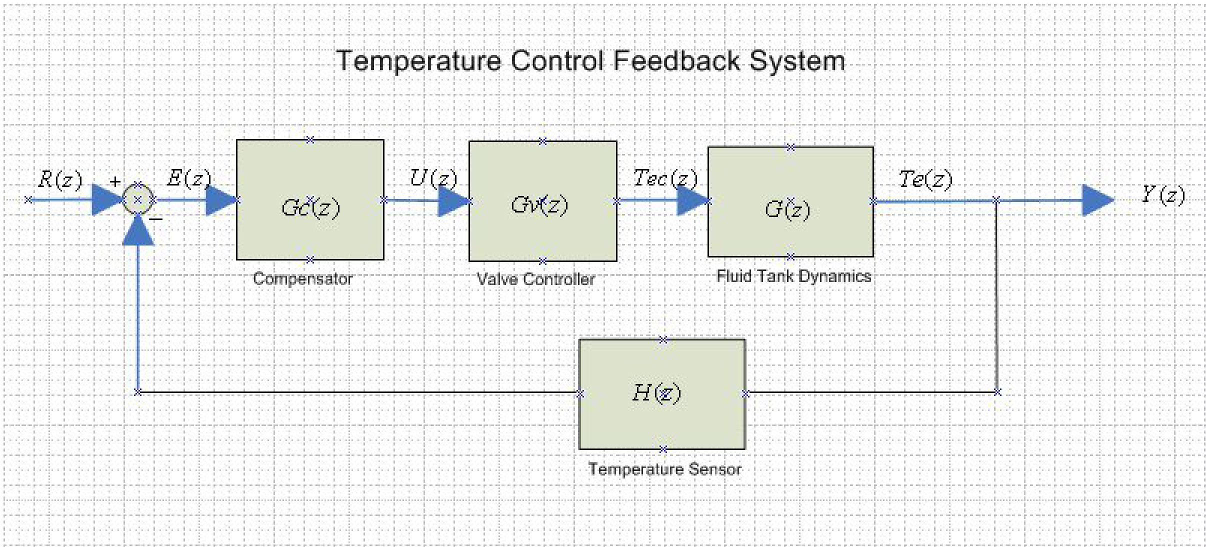

- For one more examination, we'll increase the sampling interval to $T = 5\,\mathrm{s}$
- We assume again a flow rate $m = 100\,\mathrm{kg/s}$ and compute compute the transfer functions for time delays of $7.5\,\mathrm{s}$ and $9.5\,\mathrm{s}$, respectively (corresponding to $m = .1$ and $m = .5$
- This time we obtain

$$G_5(z) = \frac{(.0488)(z + 7.0678)}{z^2(z - 0.6065)}$$
$$G_6(z) = \frac{(.2212)(z + 0.7788)}{z^2(z - 0.6065)}$$

with corresponding unit pulse responses,




- At this point, let's attempt to control the tank temperature using classical feedback control techniques
- For this, we'll first assemble a feedback control configuration for the system
  - Note that we have implicitely assumed all quantities to be sampled discrete-time




- You'll notice here that for generality, we have indicated a temperature sensor (since one must somehow exist) and a valve controller
- We won't develop this here, but we must be mindful of the practical need for a mechanism that will convert our control input signal, $U(z)$ , into a valve action that will result in the correct output temperature, $T_{ec}$, for the mixed fluid
- Note also that we have absorbed the transport delay into the tank transfer function $G(z)$
- For purposes of this case study we will focus on transfer function $G_5(z)$ :

$$G_5(z) = \frac{(.0488)(z + 7.0678)}{z^2(z - 0.6065)}$$

## *Physical Interpretation*

- Let's take a closer look at the system under study to gain a deeper understanding of its behavior

- Transfer function $G_5(z)$ relates the output (tank) fluid temperature $T_e$ to the input (control) fluid temperature $T_{ec}$, where both quanties are expressed in $^\circ K$
- Without loss of generality, we can consider
$$T_e = T - T_o$$
where $T$ is the actual tank temperature and $T_o$ is a nominal (equilibrium) state
  - In this sense, $T_e$ refers to the relative temperature difference from equilibrium and may be equivalently expressed in $^\circ C$
- Therefore, a reference demand on the system is a request to change the temperature from the current value by the amount of the reference
  - For example, $T_{ref} = 10\,C$ indicates we wish to raise the internal tank temperature by $+10\,C$
  - Conversely, $T_{ref} = -5\,C$ commands a reduction in temperature from the nominal by $5\,C$
- Our interest in the magnitude of the control effort is driven by the physical reality that places real limits on the available temperature of the hot and cold feed supplies
  - In our problem configuration, constraints placed on control magnitude reflect how much hotter (or colder) the supply temperatures must be relative to the nominal
  - We assume fluid temperature mixing takes place according to the following formula
$$T_{ec} = \gamma T_h + (1-\gamma)\,T_c$$
assuming constant specific heat and mass flow rates

– Here, $T_h$ and $T_c$ represent hot and cold supply temperatures, and $0 \leq \gamma \leq 1$ is the mixing ratio and is controlled by the mixing valve

- Our analysis below will utilize the discrete unit-step input for comparison of control performance
  – Strictly speaking, a unit-step demand requests a change in temperature of $1°C$ from the nominal value
  – Whereas this may be practical for some applications, it does not represent the full range of operation and is shown here for analysis purposes

*Proportional-Integral (PI) Control*

- The uncompensated transfer function exhibits a finite value DC gain ($G(1) = 1$) which will result in a constant offset error in steady-state
  – Additionally, modeling errors will further affect accurate set-point tracking
- For a baseline, we'll design a simple PI controller using the Ziegler-Nichols tuning rules in order to introduce integral action
- A discrete-time PI controller has the following form

$$K_{PI}(z) = K_p + \frac{K_p T z}{T_I(z-1)}$$

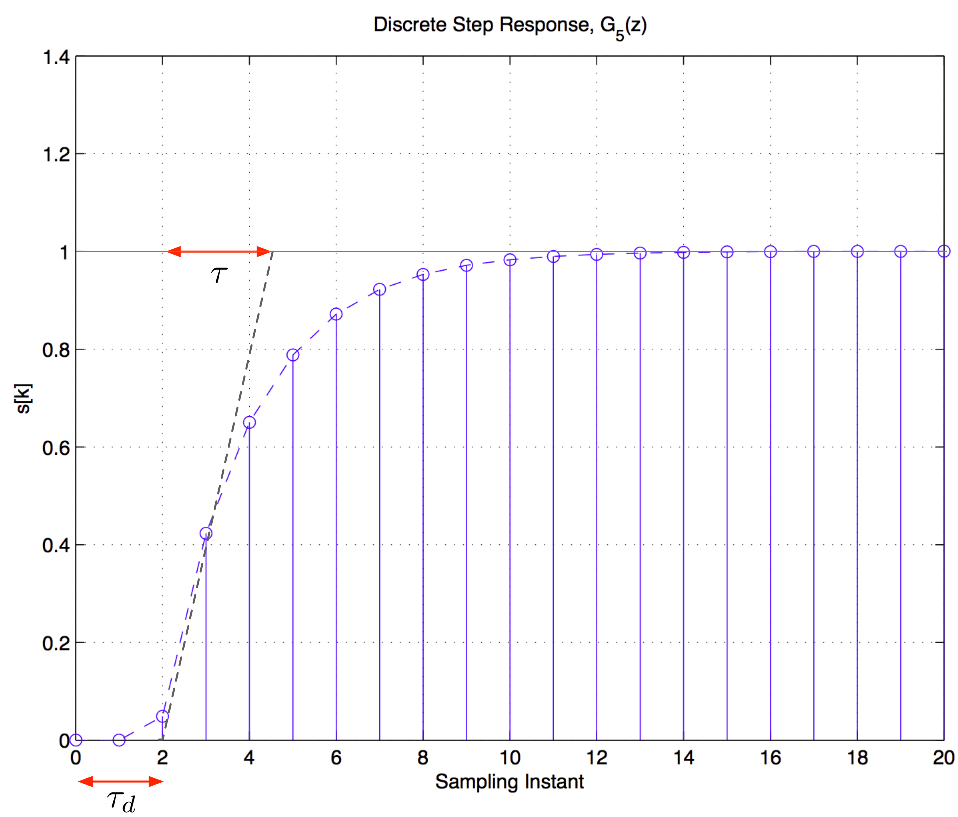
$$\begin{aligned} K_{PI}(z) &= K_p + \frac{K_p T z}{T_I(z-1)} \\ &= \frac{T_I K_p (z-1) + K_p T z}{T_I (z-1)} \end{aligned}$$

$$= \frac{(T_I + T)\,K_p z - T_I K_p}{T_I\,(z-1)}$$

$$= \frac{\frac{(T_I+T)K_p}{T_I}\left(z - \frac{T_I}{(T_I+T)}\right)}{(z-1)}$$

where we'll have to compute appropriate values for $K_p$ and $T_I$ using the tuning rules

- Our first step will be to generate an open-loop discrete-time unit step response for our plant $G_5(z)$
  - This is depicted via Matlab simulation below.




- Measuring from the plot we can determine the two process parameters $L$ and $R$ to be

$$L = \tau_d = 2.0T = 10 \text{ sec}$$
$$R = {}^{1}\!/_{\tau} = {}^{1}\!/_{2.5T} = 0.08s^{-1}$$

– From these, the Ziegler-Nichols tuning parameters are found to be

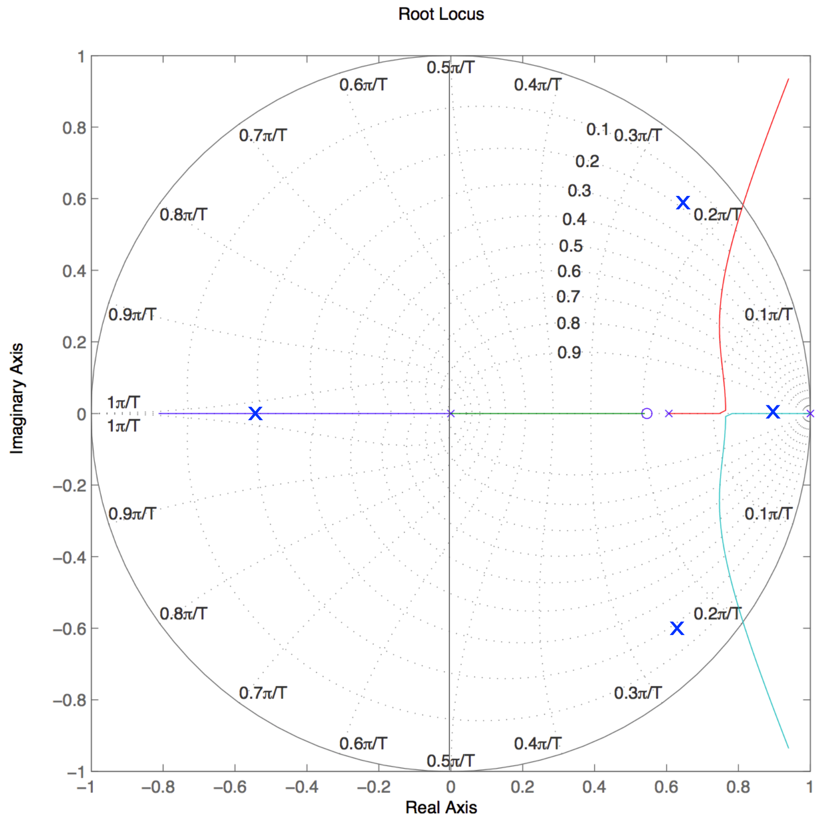
$$K_p = {}^{0.9}\!/_{RL} = {}^{0.9}\!/_{(.08)(10)} = 1.125$$
$$T_I = 3L = (3)(10) = 30.0 \text{ sec}$$

- Substituting these values gives our compensator as

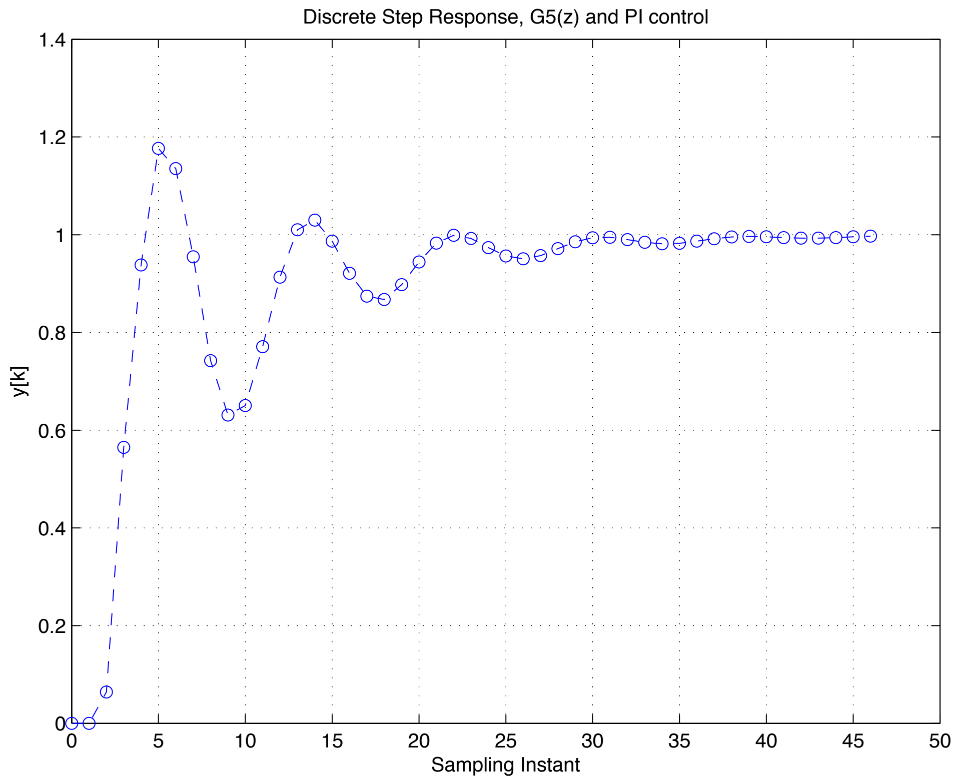
$$K_{PI}(z) = \frac{(30+5)\,(1.125)}{(30)}\left(z - \frac{30}{(30+5)}\right)$$

$$= \frac{1.3125(z - .8571)}{z-1}$$

- Here, it is useful to examine the root locus plot in order to gain some insight into closed-loop system behavior:




- The closed-loop poles are indicated by the blue ’x’s in the figure
  – Performance is governed by the complex conjugate pair of poles located at $s_{1,2} = .6356 \pm j.5941$ combined with the ’slow’ pole at $s_3 = .9031$
- Note that since the complex conjugate roots are not strictly dominant (because of the real slow pole) , we cannot directly apply the damping specifications
  – Computing the closed-loop step response of the PI-compensated system, we obtain the following:

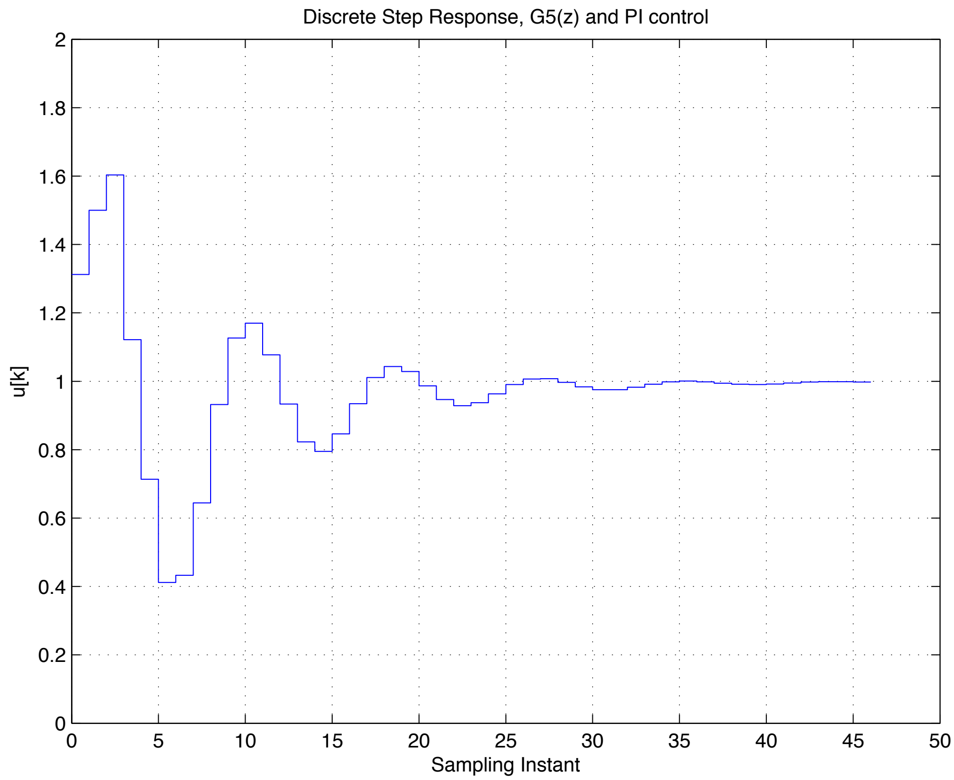



- The system exhibits an oscillatory response with a maximum peak overshoot of nearly $20\,\%$, which is typical of Ziegler-Nichols PI compensation

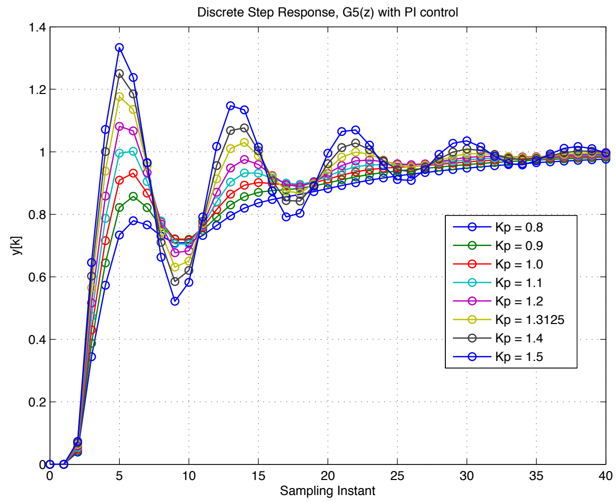
- PI compensation gives the following time-domain measures:

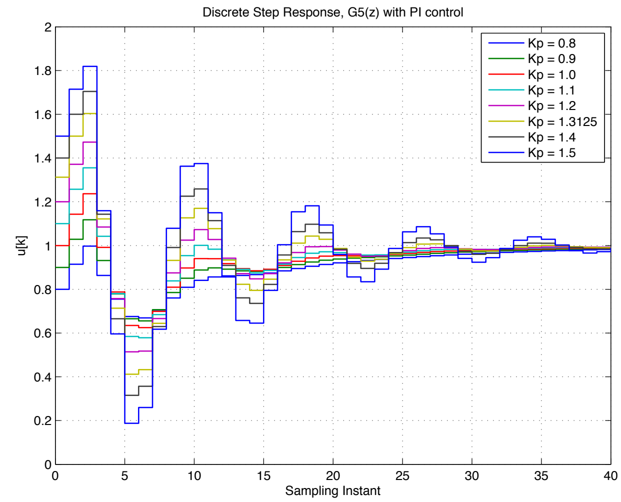
  – $t_r = 1.8\,T$ = 9.0 sec), $t_s = 35\,T$ (175 sec), $M_{PO} = 19\%$

- Examining the corresponding control magnitude, we obtain the following:




- Control effort is likewise oscillatory, with a maximum magnitude of $u_{max} = 1.6$
- Depending on the application, it may be the case that the process is sensitive to such persistent fluctuations in temperature
- Let's examine the closed-loop performance we would obtain with a range of gains $K_p$




- Clearly, by reducing the gain substantially, we can avoid excessive overshoot, but at the expense of slowing down the overall response
- So, whereas PI control eliminates steady-state error, the response is unsatisfactory in terms of excessive oscillation and slow dynamics
- With additional effort, we could extend the classical control approach with lead-lag compensation and perhaps improve on this response
- Let's now investigate what we can achieve with MPC...

## *Model Predictive Control: Unconstrained Case*

- We'll first design an unconstrained model predictive controller to see if we can improve overall system response
- Computing an equivalent state-space representation from the transfer function $G_5(z)$ we obtain

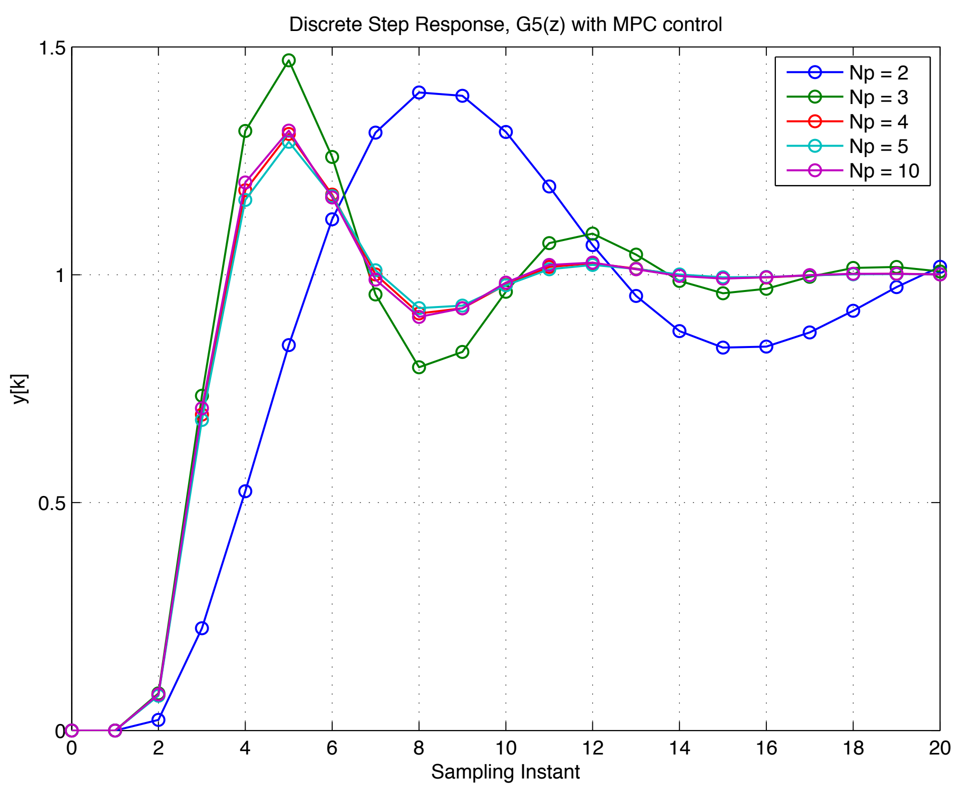
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6065 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & .0488 & 0.3743 \end{bmatrix}$$

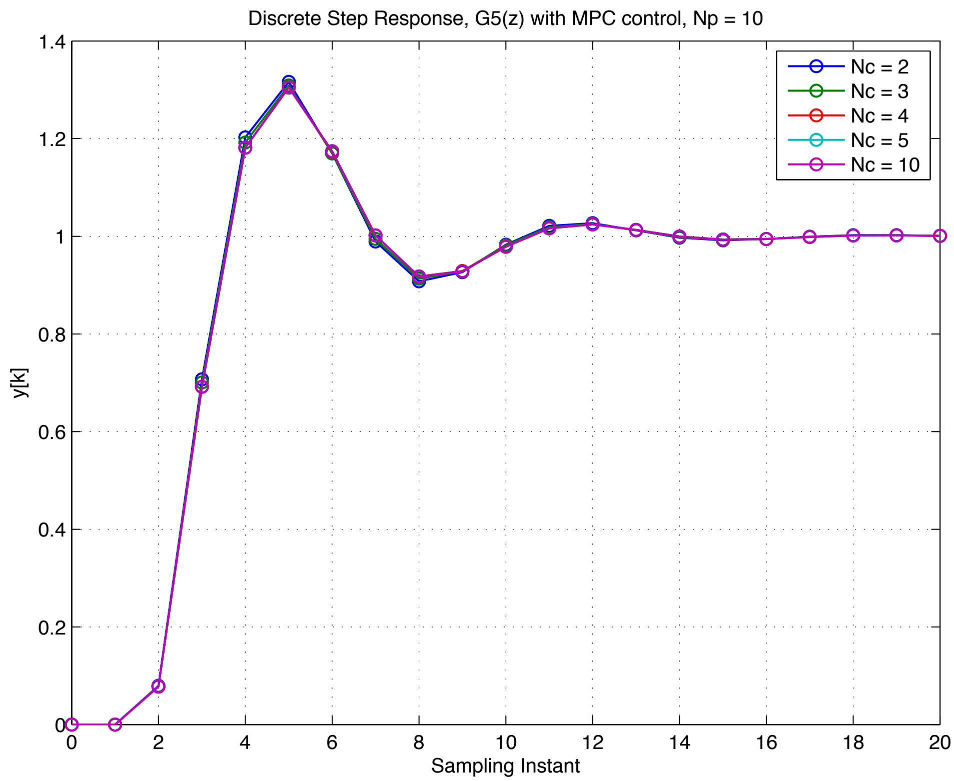
$$D = \begin{bmatrix} 0 \end{bmatrix}$$

- We'll begin by computing the MPC step response for a two-step control horizon $N_c = 2$ and range of prediction horizons between $N_p = 2$ and $N_p = 10$ while setting the control weighting $\bar{R} = 0.1$



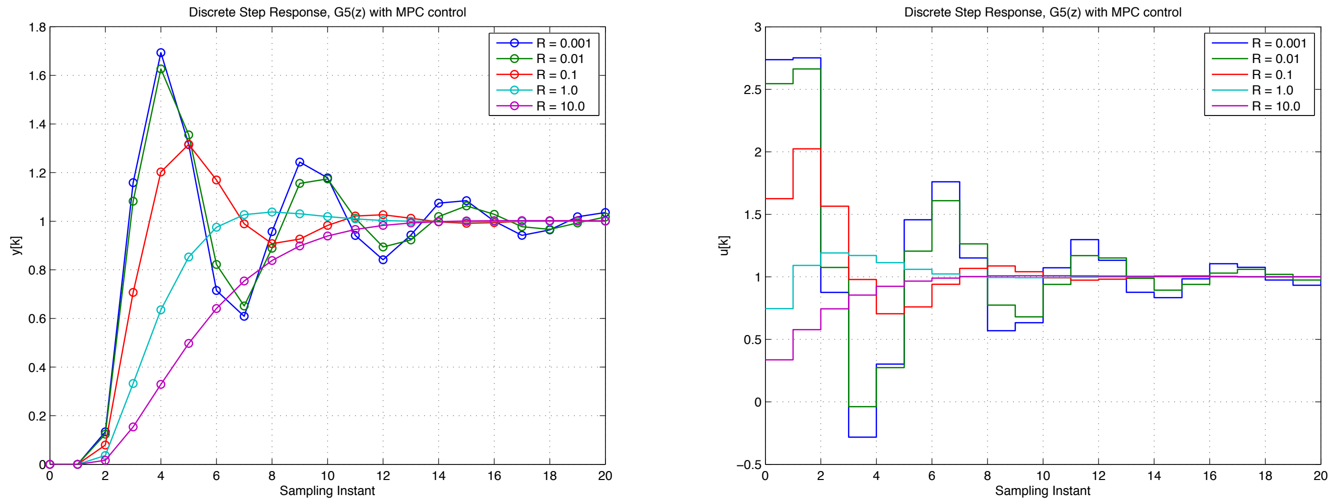

  – Responses for $N_p = 2$ and $N_p = 3$ are too oscillatory, but responses converge to a more satisfactory result for $N_p = 4$ and greater

- Let's now fix $N_p = 10$ and vary the control horizon to assess its affect on performance




  – Here we see the response is largely insensitive to values of control horizon beyond $N_c = 2$

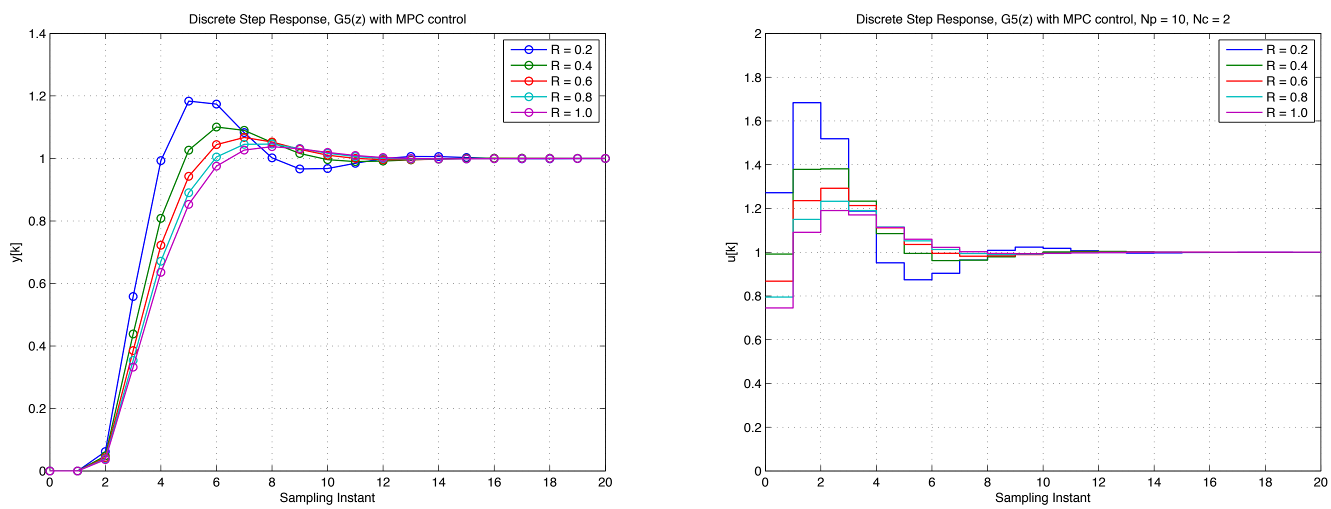
- Now, selecting values $N_p = 10$ and $N_c = 2$ , we'll vary the control weighting $\bar{R}$,




– Clearly, the output response is highly sensitive to relative amount of control weighting

– From this parametric study, it appears that good response characteristics are obtained for $0.1 \le \bar{R} \le 1.0$

– Examining this region of the parameter space more closely,

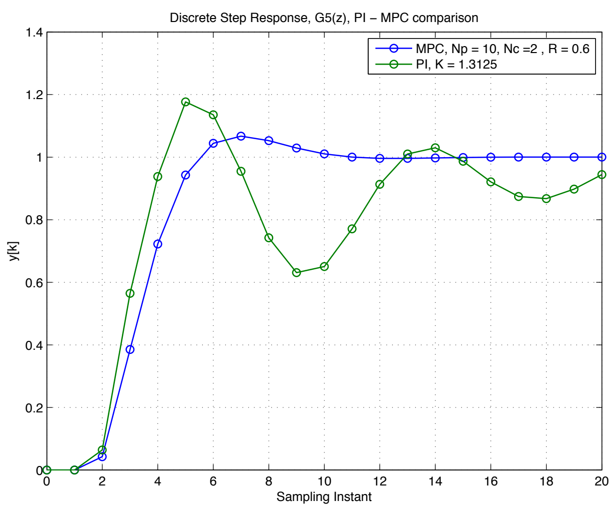


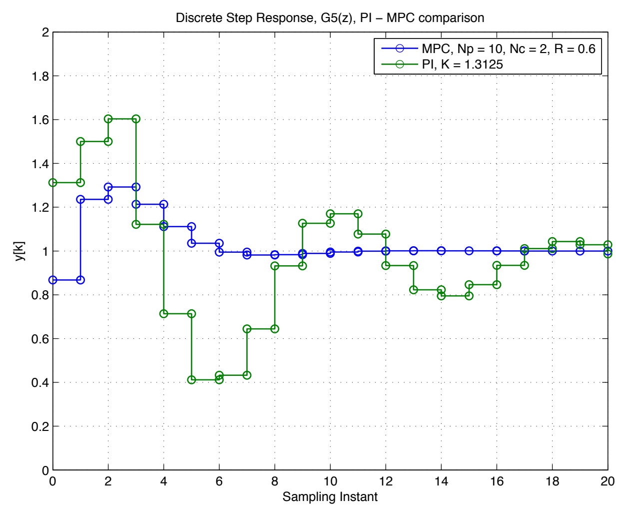

– We see that a good compromise between speed and overshoot is achieved for $\bar{R} = 0.6$

- For the selected design tuning parameters, $N_p = 10$, $N_c = 2$, and $\bar{R} = 0.6$, we obtain the following performance measures:

| Measure | MPC-uncon |
|---|---|
| $t_r$ | $13.2$ sec |
| $t_s$ | $50.0$ sec |
| $M_{PO}$ | $6.7\,\%$ |
| $u_{max}$ | $1.30$ |

- A comparison with our previous PI design appears below,




  – Comparing performance measures, we obtain:

| Measure | PI | MPC-uncon |
|---|---|---|
| $t_r$ | $9.1$ sec | $13.2$ sec |
| $t_s$ | $175$ sec | $50.0$ sec |
| $M_{PO}$ | $17.7\,\%$ | $6.7\,\%$ |
| $u_{max}$ | $1.60$ | $1.30$ |

## *Model Predictive Control: Constrained Case*

- Perhaps the greatest advantage to using MPC is its ability to enforce hard constraints on problem variables

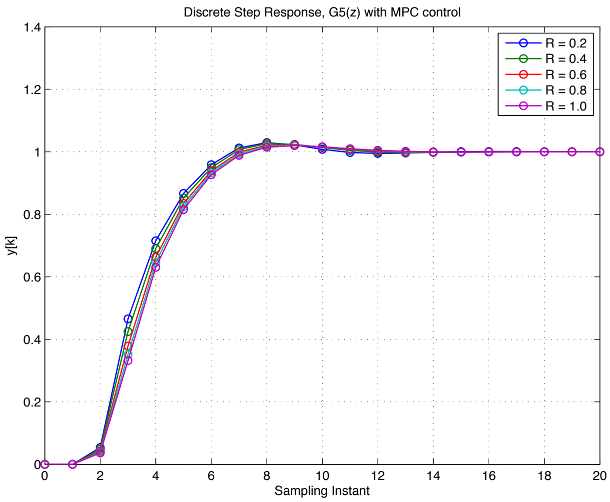
- For the given example, we shall enforce hard limits on the control magnitude such that $-1.1 \le u[k] \le 1.1$
  - The hope here is that by relaxing the control penalty - and enforcing the constraint - we can achieve a fast and acceptable response
- We introduce these bound by way of the linear constraint equation

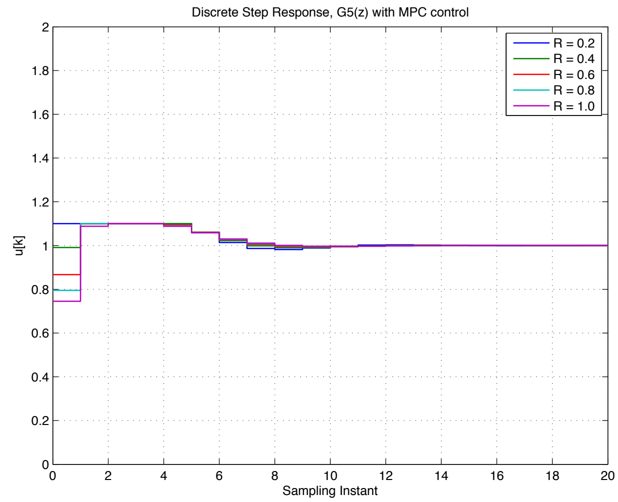
$$M\,\boldsymbol{\Delta U} \le \boldsymbol{\gamma}$$

with corresponding matrices:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ -1 & -1 \end{bmatrix}; \qquad \gamma = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 1.1 \\ 1.1 \\ -1.1 \\ -1.1 \end{bmatrix}$$

- The resulting output and control plots are presented below for the range of control weighting $0.2 \le \bar{R} \le 1.0$,



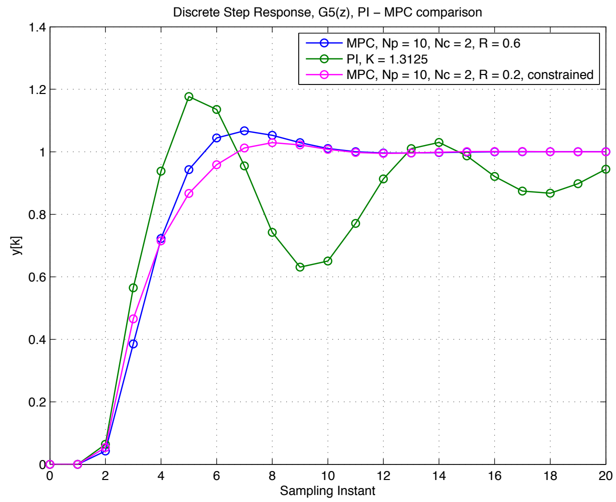



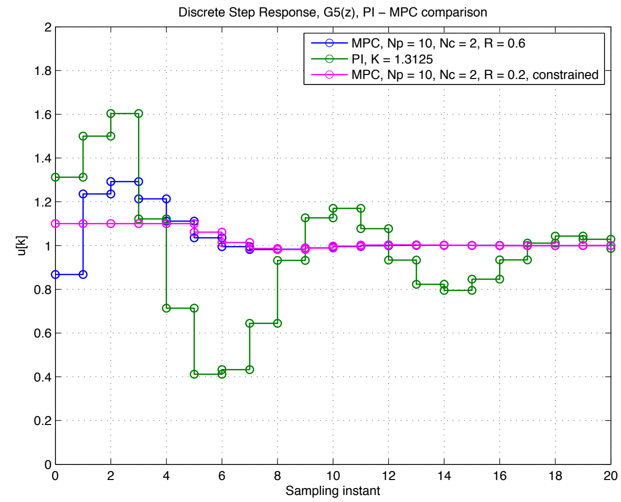

– It is clear to see that MPC held the constraint tightly, which had the effect of reducing the sensitivity of the output response to $\bar{R}$

– It is also evident we've given up some performance on account of the constraint; a comparison of the $\bar{R} = 0.6$ case with prior results appears below

| Measure | PI | MPC-uncon | MPC-con |
|---|---|---|---|
| $t_r$ | $9.1$ sec | $13.2$ sec | $17.3$ sec |
| $t_s$ | $175$ sec | $50.0$ sec | $50.0$ sec |
| $M_{PO}$ | $17.7\,\%$ | $6.7\,\%$ | $2.2\,\%$ |
| $u_{max}$ | $1.60$ | $1.30$ | $1.10$ |

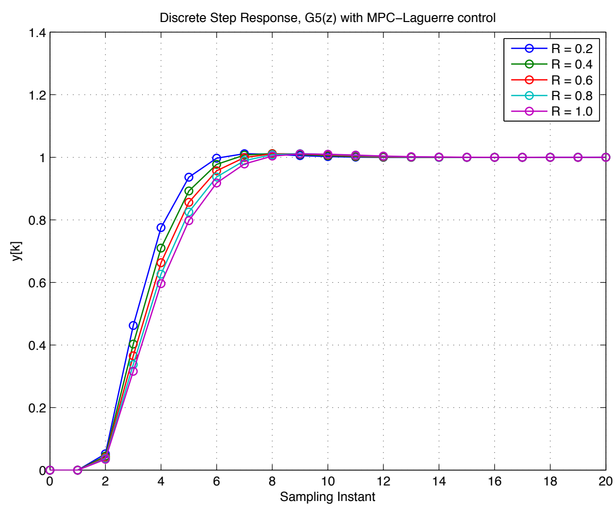
- All three cases are plotted together in the following figure

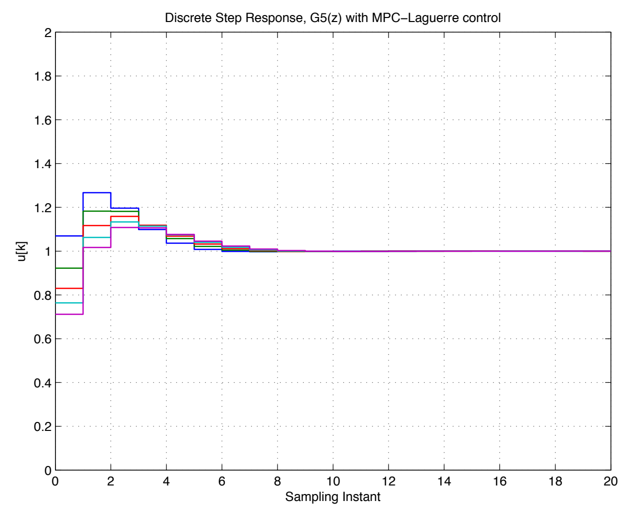



## *Model Predictive Control: Laguerre*

- Finally, we'll re-run the same case using an unconstrained version of the Laguerre expansion form of the MPC algorithm
- For this run, we'll choose the number of Laguerre functions to be $N = 3$ and select the Laguerre pole as $a = 0.8$; all other problem parameters remain the same
- The resulting output is shown below:



- For purposes of comparison, we'll compute the $\bar{R} = 0.6$ performance parameters for the Laguerre case as well,

| Measure | PI | MPC-uncon | MPC-con | MPC-Laguerre |
|---|---|---|---|---|
| $t_r$ | $9.1$ sec | $13.2$ sec | $17.3$ sec | $16.3$ sec |
| $t_s$ | $175$ sec | $50.0$ sec | $50.0$ sec | $40.0$ sec |
| $M_{PO}$ | $17.7\,\%$ | $6.7\,\%$ | $2.2\,\%$ | $1.1\,\%$ |
| $u_{max}$ | $1.60$ | $1.30$ | $1.10$ | $1.16$ |

- It appears that the Laguerre form of MPC performed on par with the best performance we obtained from constrained MPC for this example

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